Introduction to Probability Theory
Problem set #1

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Abstract

Solutions to the problem set number one of the subject Statistics, Monte Carlo Methods and Data Processing - Master in Astrophysics, Particle Physics and Cosmology (Universitat de Barcelona).

1 Problem

1.1 Individual cumulative function

Suppose we have samples for each random variable X, Y and Z (i.i.d):

$$\{X_1, ..., X_n\}\ \{Y_1, ..., Y_n\}\ \{Z_1, ..., Z_n\}$$

Let define the maximum of each sample as:

$$W_x = max\left(X_1, ..., X_n\right)$$

$$W_y = max\left(Y_1, ..., Y_n\right)$$

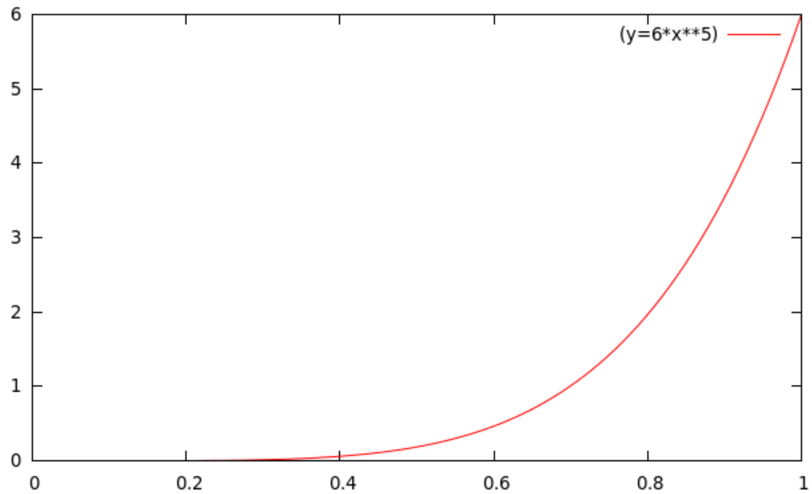
$$W_z = max\left(Z_1, ..., Z_n\right)$$

And let's consider that these maximums are ordered:

$$W_x < W_y < W_z$$

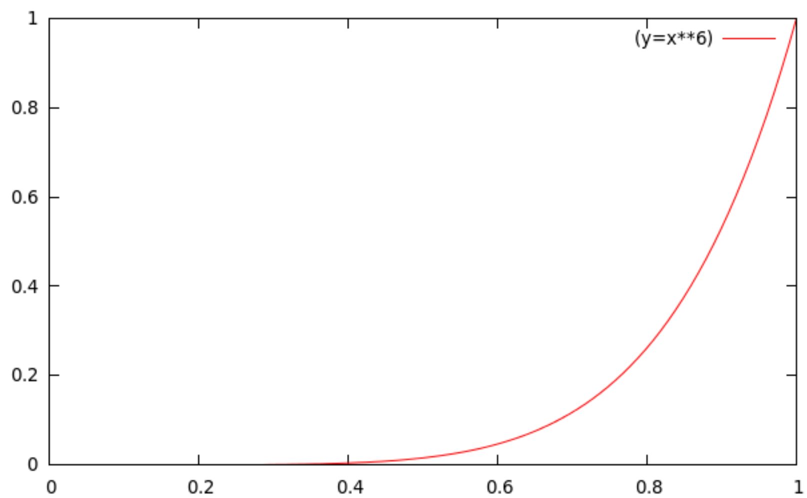
The probability distribution function of X_n, Y_n and Z_n:

$$f\left(x\right) = \begin{cases} 6x^5 & if\ 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases}$$



The cumulative distribution function of individual X_n, Y_n and Z_n:

$$F(x) = P(X \leq x) = \int_0^x 6x^5 dx = x^6 \text{ where } 0 \leq x \leq 1$$



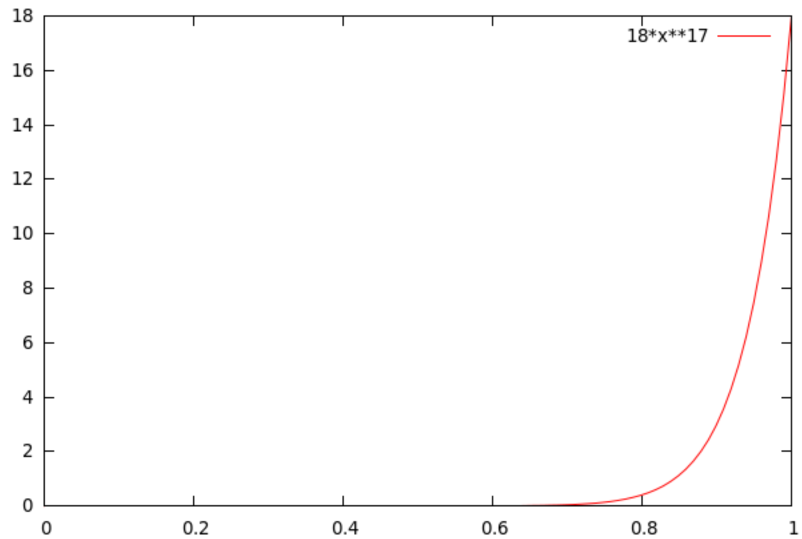
1.2 Cumulative distribution for X, Y and Z

We seek the cumulative distribution for the maximum of X, Y and Z:

$$F_x(x) = P(W_x < x) = x^6$$

$$F_{x,y}(x) = P(W_x < x, W_y < x) = x^6 \cdot x^6 = x^{12}$$

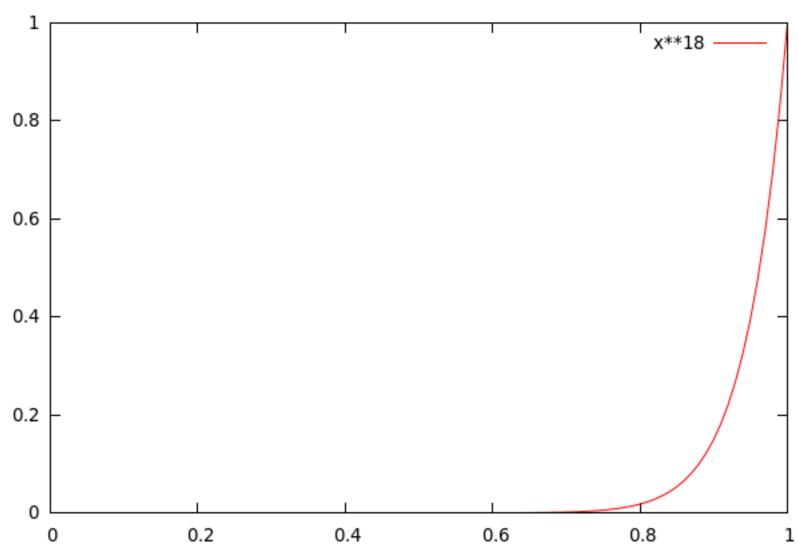
$$F_{x,y,z}(x) = P(W_x < x, W_y < x) = x^6 \cdot x^6 \cdot x^6 = x^{18}$$



1.3 Probability density of X, Y and Z

The probability density is derived from the cumulative distribution:

$$f_{x,y,z}(x) = \frac{d}{dx} F_{x,y,z}(x) = 18x^{17}$$



2 Problem

2.1 Probability of containing only two queens in the set A

Recall (combinations without repetition where order does not matter):

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Number of ways we can pick 26 cards to form set A:

- Population n: 52 cards
- Sample k: 26 cards

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{52!}{26! \cdot (52-26)!} = 495.918.532.948.104$$

Number of ways we can pick 2 queens from 4:

- Population n: 50 cards
- Sample k: 24 cards

$$\binom{4}{2} = \frac{n!}{k! \cdot (n-k)!} = \frac{4!}{2! \cdot (4-2)!} = 6$$

Number of ways we can pick 24 card from the remaining:

- Population n: 48 cards
- Sample k: 24 cards

$$\binom{48}{24} = \frac{n!}{k! \cdot (n-k)!} = \frac{48!}{24! \cdot (48-24)!} = 32.247.603.683.100$$

Probability of containing two queens in the set A:

$$P\,(\text{"}Two\, queens\, in\, set\, A\text{"}) = 6 \cdot \frac{32.247.603.683.100}{495.918.532.948.104} = \frac{325}{833} = 0.39016$$

2.2 Probability of containing the queen of spades and the queen of diamonds in the set A

Consider we have already picked the two queens and there are 50 remaining cards. Number of ways of completing the set A with 24 more cards:

- Population n: 50 cards

- Sample k: 24 cards

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{50!}{24! \cdot (50-24)!} = 121.548.660.036.300$$

Probability of containing the two queens in the set A:

$$P\left("Two\, concrete\, queens\, in\, set\, A"\right) = \frac{121.548.660.036.300}{495.918.532.948.104} = \frac{25}{102} = 0.2451$$

3 Problem

3.1 Probability density for Z = X / Y (ratio distribution)

Given:

$$X = e^{\frac{-x^2}{2\sigma_x^2}}$$

$$Y = e^{\frac{-x^2}{2\sigma_y^2}}$$

The probability density for Z:

$$Z = \frac{X}{Y} = \frac{e^{\frac{-x^2}{2\sigma_x^2}}}{e^{\frac{-x^2}{2\sigma_y^2}}} = e^{\frac{-x^2}{2\sigma_x^2} - \frac{-x^2}{2\sigma_y^2}} = e^{-\left(\frac{x^2}{2\sigma_x^2} - \frac{x^2}{2\sigma_y^2}\right)} = e^{-(\alpha)}$$

Detailed calculation of the exponent:

$$\alpha = \frac{x^2}{2\sigma_x} - \frac{x^2}{2\sigma_y} = \frac{x^2 \cdot \sigma_y - x^2 \cdot \sigma_x}{2\sigma_x^2\sigma_y^2} = \frac{x^2 \cdot (\sigma_y - \sigma_x)}{2\sigma_x^2\sigma_y^2} = \frac{x^2}{2 \cdot \frac{\sigma_x^2\sigma_y^2}{\sigma_y - \sigma_x}}$$

Therefore:

$$Z = \frac{X}{Y} = e^{-(\alpha)} = e^{-\left(\frac{x^2}{2 \cdot \frac{\sigma_x^2\sigma_y^2}{\sigma_y - \sigma_x}}\right)}$$

And it can be considered that the standard deviation of the new probability density is:

$$\sigma_z = \sqrt{\frac{\sigma_x^2\sigma_y^2}{\sigma_y - \sigma_x}}$$

In consequence, the probability density for Z can be written as:

$$Z = \frac{X}{Y} = e^{-\frac{x^2}{2\sigma_z^2}}$$

4 Problem

We've got a limited alphabet of L elements:

$$X = \{x_1, x_2, \ldots, x_L\}$$

If we choose m letters from this alphabet, each new letter is independent from the past picked letters:

$$P\left(x_n | x_{n-1}, ..., x_{n-n}\right) = P\left(x_n\right) = \frac{1}{L}$$

Therefore, individual probability for one pick (m=1 and n=1):

$$P\left(A\, letter\, being\, picked\right) = \frac{1}{L}$$

$$P\left(A\, letter\, NOT\, being\, picked\right) = 1 - \frac{1}{L}$$

And the probability of picking only 1 concrete letter (n=1) for m picks:

$$P\left(A\, letter\, being\, picked\, only\, 1\, time\right) = \left(\frac{1}{L}\right)^1 \cdot \left(1 - \frac{1}{L}\right)^{m-1}$$

Finally, the probability of picking n times a letter for m picks:

$$P\left(A\, letter\, being\, picked\, n\, times\right) = \left(\tfrac{1}{L}\right)^n \cdot \left(1 - \tfrac{1}{L}\right)^{m-n} \text{ where } n \leq m$$

5 Problem

Let's consider the following random variables (numbers correspond to the different boxes):

$SelectedBox \in \{1, 2, 3\}$My box selection
$ObjectBox \in \{1, 2, 3\}$ Box where the desired object is located
$OpenBox \in \{1, 2, 3\}$ Box opened by the Oracle

My box selection and the box where the desired object is located are independent variables:

$$P\left(SelectedBox\right) = P\left(SelectedBox | ObjectBox\right) = \frac{1}{3}$$

$$P\left(ObjectBox\right) = P\left(ObjectBox | SelectedBox\right) = \frac{1}{3}$$

Otherwise, the Oracle will open one box or another depending on my selection and on the location of the desired object:

$$P\left(OpenBox|SelectedBox, ObjectBox\right) = \begin{cases} \frac{1}{2} & if\, SelectedBox = ObjectBox \\ 1 & if\, OpenBox \neq ObjectBox \neq SelectedBox \\ 0 & if\, OpenBox = SelectedBox \\ 0 & if\, OpenBox = ObjectBox \end{cases}$$

And we wan to know, what is the probability of a box containing the desired object if we already have made an initial selection and the Oracles has open a box:

$$P\left(ObjectBox|SelectedBox, OpenBox\right)$$

To make the calculation we can use the Bayes' theorem:

Abbreviations:
$SelectedBox = Select$
$ObjectBox = Obj$
$OpenBox = Open$

$$P\left(Obj|Select, Open\right) = \frac{P\left(Open|Select, Obj\right) P\left(Obj|Select\right)}{P(Open|Select)}$$

where

$$P\left(Open|Select\right) = \sum_{Obj=1}^{3} P\left(Open, Obj|Select\right) = \sum_{ObjBox=1}^{3} P\left(Open|Select, Obj\right) P\left(Obj|Select\right)$$

For example, once I have selected a box (number 2) and the Oracle has opened number 3:

$$SelectedBox = 2$$

$$OpenBox = 3$$

The probability of the object being in the remaining box (number 1):

$$ObjectBox = 1 \neq SelectedBox$$

$$P(ObjectBox|SelectedBox, OpenBox) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{2}{3}$$

And the probability that the object is on my selected box (number 2):

$$ObjectBox = SelectedBox$$

$$P(Obj = Select|Select, Open) = 1 - P(Obj|Select, Open) = \frac{1}{3}$$

In conclusion, it is better to choose the remaining box and change my initial choice.

6 Problem

6.1 Inequality

We have four random variables (from Alice and Bob's perspective):

$\{A_1, A_2\}$ Properties that Alice is interested in
$\{B_1, B_2\}$ Properties that Bob is interested in

This random variables can take values $\{-1, +1\}$. Oscar, the oracle, draws a random variable Λ according to the probability distribution π. In this context, this variable acts as a hidden variable that determines the outcome (a_i for Alice, b_i for Bob) of any measurement:

$a_i = f_A(A_i, \lambda)$
$b_i = f_B(B_i, \lambda)$

Whereλ is a value of the random variable Λ. Given these premises, it is clear that one of the following equations must be true:

Subtraction: $f_A(A_1, \lambda) - f_A(A_2, \lambda) = 0$
Addition: $f_A(A_1, \lambda) + f_A(A_2, \lambda) = 0$

a_1	a_2	Subtraction$= 0$	Addition$= 0$
-1	-1	True	False
-1	$+1$	False	True
$+1$	$+1$	True	False
$+1$	-1	False	True

Therefore:

$$\begin{aligned} f_A(A_1, \lambda) f_B(B_1, \lambda) + f_A(A_2, \lambda) f_B(B_1, \lambda) + f_A(A_1, \lambda) f_B(B_2, \lambda) \\ -f_A(A_2, \lambda) f_B(B_2, \lambda) = f_B(B_1, \lambda)(f_A(A_1, \lambda) + f_A(A_2, \lambda)) \\ +f_B(B_2, \lambda)(f_A(A_1, \lambda) - f_A(A_2, \lambda)) \leq 2 \end{aligned} \tag{6.1.1}$$

If addition$= 0$, the first term is equal to zero and the second one cannot be greater (less) than $+2$ (-2) (i.e. maximum: $f_A(A_1,\lambda) = +1$ and $f_A(A_2,\lambda) = +1$). Otherwise, if subtraction$= 0$, the second term is equal to zero and the first one cannot be also greater (lesser) than $+2$ (-2) (i.e. maximum: $f_A(A_1,\lambda) = +1$ and $f_A(A_2,\lambda) = -1$).

After a large number of experiments, Alice and Bob can measure the correlators $\langle A_i B_j \rangle$ (the mean value of the product of their outcomes):

$\langle A_1 B_1 \rangle = \int_\Lambda f_A(A_1,\lambda)\, f_B(B_1,\lambda)\, \pi(\lambda)\, d\lambda$
$\langle A_1 B_2 \rangle = \int_\Lambda f_A(A_1,\lambda)\, f_B(B_2,\lambda)\, \pi(\lambda)\, d\lambda$
$\langle A_2 B_1 \rangle = \int_\Lambda f_A(A_2,\lambda)\, f_B(B_1,\lambda)\, \pi(\lambda)\, d\lambda$
$\langle A_2 B_2 \rangle = \int_\Lambda f_A(A_2,\lambda)\, f_B(B_2,\lambda)\, \pi(\lambda)\, d\lambda$

Now, let's consider:

$|\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle| =$
$\int_\Lambda f_A(A_1,\lambda)\, f_B(B_1,\lambda)\, \pi(\lambda)\, d\lambda + \int_\Lambda f_A(A_2,\lambda)\, f_B(B_1,\lambda)\, \pi(\lambda)\, d\lambda +$
$\int_\Lambda f_A(A_1,\lambda)\, f_B(B_2,\lambda)\, \pi(\lambda)\, d\lambda - \int_\Lambda f_A(A_2,\lambda)\, f_B(B_2,\lambda)\, \pi(\lambda)\, d\lambda$
$= \int_\Lambda \{f_A(A_1,\lambda)\, f_B(B_1,\lambda) + f_A(A_2,\lambda)\, f_B(B_1,\lambda) + f_A(A_1,\lambda)\, f_B(B_2,\lambda) -$
$f_A(A_2,\lambda)\, f_B(B_2,\lambda)\}\pi(\lambda)\, d\lambda$

Inside the integral we have the term (6.1.1) which we know it is less (greater) than $+2$ (-2), therefore:

$|\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle| \leq 2$

6.2 Joint probability distribution

Alice and Bob choose what measurement to do ($A_i$and B_i respectively) at random from some joint distribution $\psi(A_i, B_j)$, and they get their outcomes a_i and b_j. So, the whole experiment's outcome has the following probability distribution:

$$P(A_i, B_j; a_i, b_j) = \psi(A_i, B_j)\, P(a_i, b_j | A_i, B_j)$$

And outcomes a_i and b_j are determined by a hidden random variable Λ (according to the probability distribution π) with value λ and the functions $f_A(A_i,\lambda)$ and $f_B(B_j,\lambda)$. So it can be written as a convex combination of independent probability distributions and the joint probability distribution can be considered local.

6.3 Quantum mechanical experiments

As the problem stated, Alice can use two detector setting from which will measure a spin singlet property (i.e. x_A and z_A axis), and Bob can also use its own detector for measuring other two properties (i.e. axis x_B and z_B, which are rotated 135º relative to Alice's axis).

If we consider the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let's use σ^x and σ^x as the observables and the following eigenvectors:

$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle$ eigenvector of σ^z with eigenvalue $+1$
$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ eigenvector of σ^z with eigenvalue -1

Then, the spin singlet state is represented by:

$$\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, +\frac{1}{2}\right\rangle_A \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_B - \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_A \left|\frac{1}{2}, +\frac{1}{2}\right\rangle_B\right)$$

Note that products among eigenvectors correspond to tensor products $\otimes$.
Therefore, the measurements that Alice and Bob can perform:

$f_A(A_1) = \sigma^z \otimes I$
$f_A(A_2) = \sigma^x \otimes I$
$f_B(B_1) = -\frac{1}{\sqrt{2}} \cdot I \otimes (\sigma^z + \sigma^x)$
$f_B(B_2) = \frac{1}{\sqrt{2}} \cdot I \otimes (\sigma^z - \sigma^x)$

Where I is the identity matrix:

$$I = (\sigma^x)^2 \cdot (\sigma^y)^2 \cdot (\sigma^z)^2 = -i\sigma^x\sigma^y\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then we can state that:

$$\langle f_A(A_1)\, f_A(B_1)\rangle = \langle f_A(A_2)\, f_A(B_1)\rangle = \langle f_A(A_2)\, f_A(B_2)\rangle = \frac{1}{\sqrt{2}}$$

$$\langle f_A(A_1)\, f_A(B_2)\rangle = -\frac{1}{\sqrt{2}}$$

Therefore, the inequality established at the beginning of this problem is violated:
$\langle f_A(A_1)\, f_A(B_1)\rangle + \langle f_A(A_2)\, f_A(B_1)\rangle + \langle f_A(A_2)\, f_A(B_2)\rangle - \langle f_A(A_1)\, f_A(B_2)\rangle = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2$

6.4 Conclusion

The initial theoretical inequality corresponds to *Bell's theorem* and it is deduced considering that:

- Realism: Hidden variables exists (although nowadays we cannot measure them) and determine the measurements that can do Bob and Alice
- The principle of locality is respected: an object is influenced only by its near surroundings

But as quantum experiments show, this inequality is violated and therefore at least one of the above statements is false. It's implication are huge for science and physics, although it does not demonstrate that quantum physics is a complete theory, it invalidates any real-local theory which consider the above statements true.

Either instantaneous large distance influence can be possible or our universe does not have fixed properties until we measure them, which seems to be counter-intuitive ideas with a great philosophical impact about our concept of reality.

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